

On the minimal energy of tetracyclic graphs

Hongping Ma, Yongqiang Bai*

School of Mathematics and Statistics, Jiangsu Normal University,
Xuzhou 221116, China

Abstract

The energy of a graph is defined as the sum of the absolute values of the eigenvalues of its adjacency matrix. In this paper, we characterize the tetracyclic graph of order n with minimal energy. By this, the validity of a conjecture for the case $e = n + 3$ proposed by Caporossi et al. [1] has been confirmed.

Keywords: Minimal energy; Tetracyclic graph; Characteristic polynomial

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1 Introduction

Let G be a simple graph with n vertices and $A(G)$ the adjacency matrix of G . The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $A(G)$ are said to be the eigenvalues of the graph G . The energy of G is defined as

$$E = E(G) = \sum_{i=1}^n |\lambda_i|.$$

The characteristic polynomial of $A(G)$ is also called the characteristic polynomial of G , denoted by $\phi(G, x) = \det(xI - A(G)) = \sum_{i=0}^k a_i(G)x^{n-i}$. Using these coefficients of $\phi(G, x)$, the energy of G can be expressed as the Coulson integral formula [8]:

$$E(G) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{x^2} \ln \left[\left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i a_{2i}(G) x^{2i} \right)^2 + \left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i a_{2i+1}(G) x^{2i+1} \right)^2 \right] dx. \quad (1)$$

For convenience, write $b_{2i}(G) = (-1)^i a_{2i}(G)$ and $b_{2i+1}(G) = (-1)^i a_{2i+1}(G)$ for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Since the energy of a graph can be used to approximate the total π -electron energy of the molecular, it has been intensively studied. For details on graph energy, we refer to the recent book [14] and reviews [6, 7].

One of the fundamental question that is encountered in the study of graph energy is which graphs (from a given class) have minimal and maximal energies. A large of number of papers were published on such extremal problems, see Chapter 7 in [14].

*Corresponding author. E-mail addresses: hpmma@163.com (H. Ma), bmbai@163.com (Y. Bai)

A connected graph on n vertices with e edges is called an (n, e) -graph. We call an (n, e) -graph a unicyclic graph, a bicyclic graph, a tricyclic graph, and a tetracyclic graph if $e = n, n + 1, n + 2$ and $n + 3$, respectively. Follow [16], let $S_{n,e}$ be the graph obtained by the star S_n with $e - n + 1$ additional edges all connected to the same vertex, and $B_{n,e}$ be the bipartite (n, e) -graph with two vertices on one side, one of which is connected to all vertices on the other side.

In [1], Caporossi et al. gave the following conjecture:

Conjecture 1.1. [1] *Connected graphs G with $n \geq 6$ vertices, $n - 1 \leq e \leq 2(n - 2)$ edges and minimum energy are $S_{n,e}$ for $e \leq n + [(n - 7)/2]$, and $B_{n,e}$ otherwise.*

This conjecture is true when $e = n - 1, 2(n - 2)$ [1], and when $e = n$ for $n \geq 6$ [9]. Li et al. [15] showed that $B_{n,e}$ is the unique bipartite graph of order n with minimal energy for $e \leq 2n - 4$. Hou [10] proved that for $n \geq 6$, $B_{n,n+1}$ has the minimal energy among all bicyclic graphs of order n with at most one odd cycle. Let $\mathcal{G}_{n,e}$ be the set of connected graphs with n vertices and e edges. Let $\mathcal{G}_{n,e}^1$ be the subset of $\mathcal{G}_{n,e}$ which contains no disjoint two odd cycles of length p and q with $p + q \equiv 2 \pmod{4}$, and $\mathcal{G}_{n,e}^2 = \mathcal{G}_{n,e} \setminus \mathcal{G}_{n,e}^1$. Zhang and Zhou [17] characterized the graphs with minimal, second-minimal and third-minimal energy in $\mathcal{G}_{n,n+1}^1$ for $n \geq 8$. Combining the results (Lemmas 5-9) in [17] with the fact that $E(B_{n,n+1}) < E(S_{n,n+1})$ for $5 \leq n \leq 7$, we can deduce the following lemma.

Lemma 1.2. [17] *The graph with minimal energy in $\mathcal{G}_{n,n+1}^1$ is $S_{n,n+1}$ for $n = 4$ or $n \geq 8$, and $B_{n,n+1}$ for $5 \leq n \leq 7$, respectively.*

Li et al. [13] proved that $B_{n,n+2}$ has minimal energy in $\mathcal{G}_{n,n+2}^1$ for $7 \leq n \leq 9$, and for $n \geq 10$, they wanted to characterize the graphs with minimal and second-minimal energy in $\mathcal{G}_{n,n+2}^1$, but left four special graphs without determining their ordering. Huo et al. solved this problem in [11], and the results on minimal energy can be restated as follows.

Lemma 1.3. *The graph with minimal energy in $\mathcal{G}_{n,n+2}^1$ is $B_{n,n+2}$ for $7 \leq n \leq 9$ [13], and $S_{n,n+2}$ for $n \geq 10$ [11], respectively.*

In [16], the authors claimed that they gave a complete solution to conjecture 1.1 for $e = n + 1$ and $e = n + 2$ by showing the following two results.

Lemma 1.4. (Theorem 1, [16]) *Let G be a connected graph with n vertices and $n + 1$ edges. Then*

$$E(G) \geq E(S_{n,n+1})$$

with equality if and only if $G \cong S_{n,n+1}$.

Lemma 1.5. (Theorem 2, [16]) *Let G be a connected graph with n vertices and $n+2$ edges. Then*

$$E(G) \geq E(S_{n,n+2})$$

with equality if and only if $G \cong S_{n,n+2}$.

Note that $E(B_{n,n+1}) < E(S_{n,n+1})$ for $5 \leq n \leq 7$, and $E(B_{n,n+2}) < E(S_{n,n+2})$ for $6 \leq n \leq 9$. In addition, there is a little gap in the original proofs (even for large n) of Lemmas 1.4 and 1.5 in [16], respectively. For completeness, we will prove the following two results in Section 2.

Theorem 1.6. *$S_{n,n+1}$ if $n = 4$ or $n \geq 8$, $B_{n,n+1}$ if $5 \leq n \leq 7$ has minimal energy in $\mathcal{G}_{n,n+1}$.*

Theorem 1.7. *The complete graph K_4 if $n = 4$, $S_{n,n+2}$ if $n = 5$ or $n \geq 10$, $B_{n,n+2}$ if $6 \leq n \leq 9$ has minimal energy in $\mathcal{G}_{n,n+2}$. Furthermore, $S_{6,8}$ has second-minimal energy in $\mathcal{G}_{6,8}$.*

Li and Li [12] discussed the graph with minimal energy in $\mathcal{G}_{n,n+3}^1$, and claimed that the graph with minimal energy in $\mathcal{G}_{n,n+3}^1$ is $B_{n,n+3}$ for $9 \leq n \leq 17$, and $S_{n,n+3}$ for $n \geq 18$, respectively. Note that $E(S_{n,n+3}) < E(B_{n,n+3})$ for $n \geq 12$. In Section 3, we will first illustrate the correct version of this result, and then we will show the following theorem.

Theorem 1.8. *The wheel graph W_5 if $n = 5$, the complete bipartite graph $K_{3,3}$ if $n = 6$, $B_{n,n+3}$ if $7 \leq n \leq 11$, $S_{n,n+3}$ if $n \geq 12$ has minimal energy in $\mathcal{G}_{n,n+3}$. Furthermore, $S_{n,n+3}$ has second-minimal energy in $\mathcal{G}_{n,n+3}$ for $6 \leq n \leq 7$.*

Lemma 1.9. [16] *$E(S_{n,e}) < E(B_{n,e})$ if $n-1 \leq e \leq \frac{3}{2}n-3$; $E(B_{n,e}) < E(S_{n,e})$ if $\frac{3}{2}n - \frac{5}{2} \leq e \leq 2n-4$.*

From Lemma 1.9, we know that the bound $e \leq n + \lceil (n-7)/2 \rceil$ in Conjecture 1.1 should be understood that $e \leq n + \lceil (n-7)/2 \rceil$. With Theorems 1.6, 1.7 and 1.8, we give a complete solution to Conjecture 1.1 for $e = n+1, n+2$ and $n+3$.

2 The graphs with minimal energy in $\mathcal{G}_{n,n+1}$ and $\mathcal{G}_{n,n+2}$

The following three lemmas are need in the sequel.

Lemma 2.1. [5] *If F is an edge cut of a simple graph G , then $E(G-F) \leq E(G)$, where $G-F$ is the subgraph obtained from G by deleting the edges in F .*

Lemma 2.2. [16] (1) Suppose that $n_1, n_2 \geq 3$ and $n = n_1 + n_2$. Then

$$E(S_{n_1, n_1} \cup S_{n_2, n_2}) \geq E(S_{n-3, n-3} \cup C_3)$$

with equality if and only if $\{n_1, n_2\} = \{3, n-3\}$.

(2) $E(S_{n-3, n-3} \cup C_3) > E(S_{n, n+1})$ for $n \geq 6$.

(3) $E(S_{n, n+1}) > E(S_{n, n})$ for $n \geq 4$.

(4) $E(S_{n-3, n-3} \cup C_3) > E(S_{n, n+2})$ for $n \geq 6$.

Lemma 2.3. (1) [9] $S_{n, n}$ has minimal energy in $\mathcal{G}_{n, n}$ for $n = 3$ or $n \geq 6$.

(2) $B_{n, n}$ and $S_{n, n}$ have, respectively, minimal and second-minimal energy in $\mathcal{G}_{n, n}$ for $4 \leq n \leq 5$. In particular, $S_{n, n}$ is the unique non-bipartite graph in $\mathcal{G}_{n, n}$ with minimal energy for $4 \leq n \leq 5$.

Proof. By Table 1 of [3], there are two (4, 4)-graphs and five (5, 5)-graphs. By simple computation, we can obtain the result (2). ■

Proof of Theorem 1.6: By Lemma 1.2, it suffices to prove that $E(G) > E(S_{n, n+1})$ when $n = 4$ or $n \geq 8$, and $E(G) > E(B_{n, n+1})$ when $5 \leq n \leq 7$ for $G \in \mathcal{G}_{n, n+1}^2$.

Suppose that $G \in \mathcal{G}_{n, n+1}^2$. As there is nothing to prove for the case $n \leq 5$, we suppose that $n \geq 6$. Then G has a cut edge f such that $G - f$ contains exactly two components, say G_1 and G_2 , which are non-bipartite unicyclic graphs. Let $|V(G_1)| = n_1$, $|V(G_2)| = n_2$, and $n_1 + n_2 = n$. By Lemmas 2.1, 2.2 and 2.3, we have

$$E(G) \geq E(G_1 \cup G_2) \tag{2}$$

$$\geq E(S_{n_1, n_1} \cup S_{n_2, n_2}) \tag{3}$$

$$\geq E(S_{n-3, n-3} \cup C_3) \tag{4}$$

$$> E(S_{n, n+1}). \tag{5}$$

In particular, $E(G) > E(S_{n, n+1}) > E(B_{n, n+1})$ for $6 \leq n \leq 7$. The proof is thus complete. ■

Remark 2.4. The proof of Theorem 1.6 (for large n) is similar to that of Lemma 1.4 except that in [16], the authors did not point out that G_1 and G_2 are non-bipartite unicyclic graphs. Without this assumption, we know that the inequality (3) does not hold when n_1 or n_2 equals to 4 or 5 by Lemma 2.3 (2). Moreover, the inequality $E(G_1 \cup G_2) \geq E(S_{n-3, n-3} \cup C_3)$ does not hold. For example: $E(C_4 \cup S_{n-4, n-4}) < E(S_{n-3, n-3} \cup C_3)$ for $n \geq 7$, since $E(C_4) = E(C_3) = 4$ and $E(S_{n-4, n-4}) < E(S_{n-3, n-3})$ by Lemma 2.1.

Lemma 2.5. $S_{n, n+1}$ is the unique non-bipartite graph in $\mathcal{G}_{n, n+1}$ with minimal energy for $5 \leq n \leq 7$. Furthermore, $S_{n, n+1}$ has second-minimal energy in $\mathcal{G}_{n, n+1}$ for $n = 5$ or 7, and $S_{6, 7}$ has third-minimal energy in $\mathcal{G}_{6, 7}$.

Proof. By Table 1 of [3], there are five $(5, 6)$ -graphs. By simple calculation, we can prove the theorem for $n = 5$. By Table 1 of [4], there are 19 $(6, 7)$ -graphs. By direct computation, we can prove the theorem for $n = 6$. By the results (Lemmas 5-9) in [17], we can obtain that $S_{7,8}$ has second-minimal energy in $\mathcal{G}_{7,8}^1$. On the other hand, from the proof of Theorem 1.6, $E(G) > E(S_{7,8})$ for $G \in \mathcal{G}_{7,8}^2$. Therefore $S_{7,8}$ has second-minimal energy in $\mathcal{G}_{7,8}$, and so the theorem is true for $n = 7$. ■

Proof of Theorem 1.7: Since K_4 is the unique graph in $\mathcal{G}_{4,6}$, the theorem holds for $n = 4$. By Table 1 of [3], there are four $(5, 7)$ -graphs. By simple calculation, we can prove the theorem for $n = 5$. By Table 1 of [4], there are 22 $(6, 8)$ -graphs. By direct computation, we can prove the theorem for $n = 6$. Now suppose that $n \geq 7$. By Lemma 1.3, it suffices to prove that $E(G) > E(S_{n,n+2})$ when $n \geq 10$, and $E(G) > E(B_{n,n+2})$ when $7 \leq n \leq 9$ for $G \in \mathcal{G}_{n,n+2}^2$.

Suppose that $G \in \mathcal{G}_{n,n+2}^2$ and C_p, C_q are two disjoint odd cycles with $p + q \equiv 2 \pmod{4}$. Then there are at most two edge disjoint paths in G connecting C_p and C_q .

Case 1. There exists exactly an edge disjoint path P connecting C_p and C_q . Then there exists an edge e of P such that $G - e = G_1 \cup G_2$, where G_1 is a non-bipartite bicyclic graph with $n_1 \geq 4$ vertices and G_2 is a non-bipartite unicyclic graph with $n_2 \geq 3$ vertices. By Lemmas 2.1, 2.2, 2.3, 2.5 and Theorem 1.6, we have

$$\begin{aligned} E(G) &\geq E(G_1 \cup G_2) \geq E(S_{n_1, n_1+1} \cup S_{n_2, n_2}) > E(S_{n_1, n_1} \cup S_{n_2, n_2}) \\ &\geq E(S_{n-3, n-3} \cup C_3) > E(S_{n, n+2}). \end{aligned}$$

In particular, $E(G) > E(S_{n, n+2}) > E(B_{n, n+2})$ for $7 \leq n \leq 9$.

Case 2. There exist exactly two edge disjoint paths P^1 and P^2 connecting C_p and C_q . Then there exist two edges e_1 and e_2 such that e_i is an edge of P^i for $i = 1, 2$, and $G - \{e_1, e_2\} = G_3 \cup G_4$, where G_3 and G_4 are non-bipartite unicyclic graphs. Let $|V(G_3)| = n_1$ and $|V(G_4)| = n_2$. Then by Lemmas 2.1, 2.2 and 2.3, we have

$$E(G) \geq E(G_3 \cup G_4) \geq E(S_{n_1, n_1} \cup S_{n_2, n_2}) \geq E(S_{n-3, n-3} \cup C_3) > E(S_{n, n+2}).$$

In particular, $E(G) > E(S_{n, n+2}) > E(B_{n, n+2})$ for $7 \leq n \leq 9$. The proof is thus complete. ■

Remark 2.6. The proof of Theorem 1.7 (for large n) is similar to that of Lemma 1.5 except that in [16], the authors did not point out that G_1 and G_2 are non-bipartite graphs.

3 The graph with minimal energy in $\mathcal{G}_{n, n+3}$

Li and Li [12] discussed the graph with minimal energy in $\mathcal{G}_{n, n+3}^1$, and we first restate their results.

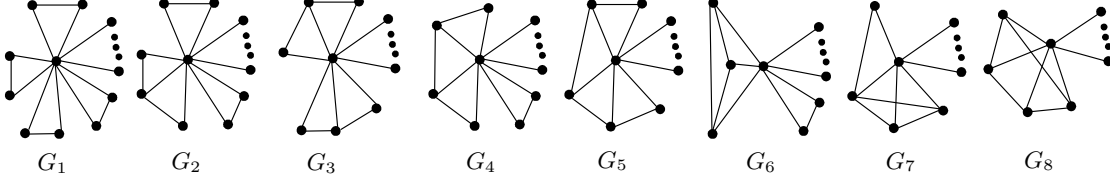


Figure 1: Graphs $G_1, G_2, G_3, G_4, G_5, G_6, G_7$ and G_8 .

Follow [12], let G_1, G_2, \dots, G_8 be eight special graphs in $\mathcal{G}_{n,n+3}$ as shown in Figure

1. Let $\mathcal{J}_n = \{S_{n,n+3}, B_{n,n+3}, G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8\}$.

Lemma 3.1. [12] *If $G \in \mathcal{G}_{n,n+3}^1$ and $G \notin \mathcal{J}_n$, then $E(G) > E(B_{n,n+3})$ for $n \geq 9$.*

In fact, Lemma 3.1 is also true for $n = 8$.

Lemma 3.2. *If $G \in \mathcal{G}_{8,11}^1$ and $G \notin \mathcal{J}_8 \setminus \{G_1\}$, then $E(G) > E(B_{8,11})$.*

Proof. By the results (see the proofs of Lemma 2.2 and Proposition 2.3) of [12], all we need is to show that $b_4(G) - b_4(B_{8,11}) > 0$ when G contains exactly i ($i = 10, 12, 13, 14, 15$) cycles (see Case 7 of Lemma 2.2). From [12], we have

$$b_4(G) - b_4(B_{8,11}) \geq \frac{1}{2}n^2 + \frac{3}{2}n - 12 - 2s - (5n - 35),$$

where s is the number of quadrangles in G . It is easy to check that in this case, G has at most 13 quadrangles. Therefore

$$b_4(G) - b_4(B_{8,11}) \geq \frac{1}{2}n^2 + \frac{3}{2}n - 12 - 26 - (5n - 35) = \frac{1}{2}n(n - 7) - 3 = 1 > 0.$$

The proof is thus complete. ■

From Lemma 1.9, we can obtain the following result.

Corollary 3.3. *$E(S_{n,n+3}) < E(B_{n,n+3})$ for $n \geq 12$, and $E(B_{n,n+3}) < E(S_{n,n+3})$ for $7 \leq n \leq 11$.*

In [12], the authors failed to get the above result in that (in the proof of Proposition 2.5 of [12]) they used the wrong formula $b_4(S_{n,n+3}) = 4n - 18$ instead of the correct one $b_4(S_{n,n+3}) = 4n - 24$. They also gave the following result.

Lemma 3.4. [12] *For each $G_j \in \mathcal{J}_n$ ($j = 1, \dots, 8$), $E(S_{n,n+3}) < E(G_j)$ for $n \geq 9$ and $E(B_{n,n+3}) < E(G_j)$ for $9 \leq n \leq 17$.*

By the proof of Lemma 2.4 of [12], we can get the following result for $n = 8$.

Lemma 3.5. *For each $G_j \in \mathcal{J}_n \setminus \{G_1\}$ ($j = 2, \dots, 8$), $E(B_{n,n+3}) < E(G_j)$ for $n = 8$.*

By Lemmas 3.1, 3.2, 3.4, 3.5 and Corollary 3.3, we can characterize the graph with minimal energy in $\mathcal{G}_{n,n+3}^1$.

Lemma 3.6. *The graph with minimal energy in $\mathcal{G}_{n,n+3}^1$ is $B_{n,n+3}$ for $8 \leq n \leq 11$, and $S_{n,n+3}$ for $n \geq 12$, respectively.*

To prove Theorem 1.8, we need the following two lemmas.

Lemma 3.7. (1) $E(K_4) > E(S_{4,4})$, and $E(B_{n,n+2}) > E(S_{n,n})$ for $7 \leq n \leq 9$.

(2) $E(S_{n,n+2}) > E(S_{n,n})$ for $n \geq 5$.

Proof. (1) It is easy to obtain that $E(K_4) = 6$, $E(S_{4,4}) \doteq 4.96239$, $E(B_{7,9}) \doteq 7.21110$, $E(S_{7,7}) \doteq 6.64681$, $E(B_{8,10}) \doteq 7.91375$, $E(S_{8,8}) \doteq 7.07326$, $E(B_{9,11}) \doteq 8.46834$ and $E(S_{9,9}) \doteq 7.46410$. Hence the result (1) follows.

(2) Since $6 = E(S_{5,7}) > E(S_{5,5}) \doteq 5.62721$, we now suppose $n \geq 6$. By direct computation, we have that $\phi(S_{n,n+2}, x) = x^n - (n+2)x^{n-2} - 6x^{n-3} + (3n-15)x^{n-4}$ and $\phi(S_{n,n}, x) = x^n - nx^{n-2} - 2x^{n-3} + (n-3)x^{n-4}$. By Eq. (1), we obtain that

$$\begin{aligned} E(S_{n,n+2}) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{x^2} \ln((1 + (n+2)x^2 + (3n-15)x^4)^2 + (6x^3)^2) dx \\ &> \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{x^2} \ln((1 + nx^2 + (n-3)x^4)^2 + (2x^3)^2) dx \\ &= E(S_{n,n}). \end{aligned}$$

■

Lemma 3.8. $E(S_{n-3,n-3} \cup C_3) > E(S_{n,n+3})$ for $n \geq 6$.

Proof. For $6 \leq n \leq 14$, the result follows by direct computation. Suppose that $n \geq 15$. By direct calculation, we have that $\phi(S_{n,n+3}, x) = x^n - (n+3)x^{n-2} - 8x^{n-3} + (4n-24)x^{n-4}$. Let $f(x) = x^4 - (n+3)x^2 - 8x + 4n-24$. Then we have that $f(-\sqrt{n-1}) > 0$, $f(-2) < 0$, $f(0) > 0$, $f(2) < 0$ and $f(\sqrt{n+3}) > 0$. Hence

$$E(S_{n,n+3}) < 4 + \sqrt{n-1} + \sqrt{n+3}.$$

On the other hand, we have $E(S_{n-3,n-3} \cup C_3) > 4 + \sqrt{2} + 2\sqrt{n-4}$ [16], and so $E(S_{n-3,n-3} \cup C_3) > E(S_{n,n+3})$. ■

Proof of Theorem 1.8: By Table 1 of [3], there are two (5, 8)-graphs. By simple calculation, we can prove the theorem for $n = 5$. By Table 1 of [4], there are 20 (6, 9)-graphs. By direct computation, we can prove the theorem for $n = 6$. By [2], there are 132 (7, 10)-graphs. By direct computing, we can prove the theorem for $n = 7$. Now suppose that $n \geq 8$. By Lemma 3.6 and Corollary 3.3, it suffices to prove that $E(G) > E(S_{n,n+3})$ for $G \in \mathcal{G}_{n,n+3}^2$.

Suppose that $G \in \mathcal{G}_{n,n+3}^2$ and C_p, C_q are two disjoint odd cycles with $p + q \equiv 2 \pmod{4}$. Then there are at most three edge disjoint paths in G connecting C_p and C_q .

Case 1. There exists exactly an edge disjoint path P^1 connecting C_p and C_q . Then there exists an edge e_1 of P^1 such that $G - e_1 = G_1 \cup G_2$, where either both G_1 and G_2 are non-bipartite bicyclic graphs, or G_1 is a non-bipartite tricyclic graph and G_2 is a non-bipartite unicyclic graph. Let $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$.

Subcase 1.1. Both G_1 and G_2 are non-bipartite bicyclic graphs. Then by Lemmas 2.1, 2.2, 2.5, 3.8 and Theorem 1.6, we have

$$\begin{aligned} E(G) &\geq E(G_1 \cup G_2) \geq E(S_{n_1, n_1+1} \cup S_{n_2, n_2+1}) > E(S_{n_1, n_1} \cup S_{n_2, n_2}) \\ &\geq E(S_{n-3, n-3} \cup C_3) > E(S_{n, n+3}). \end{aligned}$$

Subcase 1.2. G_1 is a non-bipartite tricyclic graph and G_2 is a non-bipartite unicyclic graph. It follows from Theorem 1.7 and Lemma 3.7 that $E(G_1) > E(S_{n_1, n_1})$. Therefore by Lemmas 2.1, 2.2, 2.3 and 3.8, we have

$$E(G) \geq E(G_1 \cup G_2) > E(S_{n_1, n_1} \cup S_{n_2, n_2}) \geq E(S_{n-3, n-3} \cup C_3) > E(S_{n, n+3}).$$

Case 2. There exist exactly two edge disjoint paths P^2 and P^3 connecting C_p and C_q . Then there exist two edges e_2 and e_3 such that e_i is an edge of P^i for $i = 2, 3$, and $G - \{e_2, e_3\} = G_3 \cup G_4$, where G_3 is a non-bipartite bicyclic graph with n_1 vertices and G_4 is a non-bipartite unicyclic graph with n_2 vertices. By Lemmas 2.1, 2.2, 2.3, 2.5, 3.8 and Theorem 1.6, we have

$$\begin{aligned} E(G) &\geq E(G_3 \cup G_4) \geq E(S_{n_1, n_1+1} \cup S_{n_2, n_2}) > E(S_{n_1, n_1} \cup S_{n_2, n_2}) \\ &\geq E(S_{n-3, n-3} \cup C_3) > E(S_{n, n+3}). \end{aligned}$$

Case 3. There exist exactly three edge disjoint paths P^4, P^5 and P^6 connecting C_p and C_q . Then there exist three edges e_4, e_5 and e_6 such that e_i is an edge of P^i for $i = 4, 5, 6$, and $G - \{e_4, e_5, e_6\} = G_5 \cup G_6$, where G_5 and G_6 are non-bipartite unicyclic graphs. Let $|V(G_5)| = n_1$ and $|V(G_6)| = n_2$. Then by Lemmas 2.1, 2.2, 2.3 and 3.8, we have

$$E(G) \geq E(G_5 \cup G_6) \geq E(S_{n_1, n_1} \cup S_{n_2, n_2}) \geq E(S_{n-3, n-3} \cup C_3) > E(S_{n, n+3}).$$

The proof is thus complete. ■

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References

- [1] G. Caporossi, D. Cvetković, I. Gutman, P. Hansen, Variable neighborhood search for extremal graphs. 2. Finding graphs with extremal energy, *J. Chem. Inf. Comput. Sci.* 39 (1999) 984–996.
- [2] D. Cvetković, M. Doob, I. Gutman, A. Torgašev, *Recent Results in the Theory of Graph Spectra*, North-Holland, Amsterdam, 1988.
- [3] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Application*, Academic Press, New York, 1980.
- [4] D. Cvetković, M. Petrić, A table of connected graphs on six vertices, *Discrete Math.* 50 (1984), 37–49.
- [5] J. Day, W. So, Graph energy change due to edge deletion, *Linear Algebra Appl.* 428 (2008) 2070–2078.
- [6] I. Gutman, The Energy of a Graph: Old and New Results, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), *Algebraic Combinatorics and Applications*, Springer-Verlag, Berlin, 2001, pp. 196–211.
- [7] I. Gutman, X. Li, J. Zhang, Graph Energy, in: M. Dehmer, F. Emmert-Streib (Eds.), *Analysis of Complex Networks: From Biology to Linguistics*, Wiley-VCH, Weinheim, 2009, 145–174.
- [8] I. Gutman, O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer-Verlag, Berlin, 1986.
- [9] Y. Hou, Unicyclic graphs with minimal energy, *J. Math. Chem.* 29 (2001) 163–168.
- [10] Y. Hou, Bicyclic graphs with minimum energy, *Linear Multilinear Algebra* 49 (2001) 347–354.
- [11] B. Huo, S. Ji, X. Li, Solutions to unsolved problems on the minimal energies of two classes of graphs, *MATCH Commun. Math. Comput. Chem.* 66 (2011) 943–958.
- [12] S. Li, X. Li, On tetracyclic graphs with minimal energy, *MATCH Commun. Math. Comput. Chem.* 60 (2008) 395–414.
- [13] S. Li, X. Li, Z. Zhu, On tricyclic graphs with minimal energy, *MATCH Commun. Math. Comput. Chem.* 59 (2008) 397–419.
- [14] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.
- [15] X. Li, J. Zhang, L. Wang, On bipartite graphs with minimal energy, *Discrete Appl. Math.* 157 (2009) 869–873.
- [16] J. Zhang, H. Kan, On the minimal energy of graphs, *Linear Algebra Appl.* 453 (2014) 141–153.
- [17] J. Zhang, B. Zhou, On bicyclic graphs with minimal energies, *J. Math. Chem.* 37 (4) (2005) 423–431.